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Statistics of lattice animals

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Abstract. We investigate the large-n behaviour of the number of lattice animals with n vertices having α cycles per vertex. We prove concavity and continuity properties for the corresponding growth constant and, in particular, show that lattice trees are exponentially scarce in the set of lattice animals. We also consider the corresponding generating function and prove a number of theorems which bound it and set other limits on its possible behaviour.

1. Introduction

In the same way that self-avoiding walks have become a standard model of excluded-volume effects in linear polymers in dilute solution, lattice animals are now an accepted model of excluded-volume effects in branched polymers. Although lattice animals had previously been studied in the mathematical literature (see, e.g., Klarner 1967) they were first seriously considered as a model of polymers by Lubensky and Isaacson (1979). A lattice animal is a connected subgraph of the lattice and we write a_n for the number of animals (up to translation) with n vertices. For the square lattice, it is easy to see that $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, $a_4 = 23$, etc. We can also consider the number of trees with n vertices, $a_n(0)$, which are the subset of animals having no cycles. Again, for the square lattice, $a_1(0) = 1$, $a_2(0) = 2$, $a_3(0) = 6$, $a_4(0) = 22$, etc (see, e.g., Whittington *et al* (1983) for more extensive tables). We can generalise this by defining the number of c animals, $a_n(c)$, to be the number of animals with n vertices and cyclomatic index c, so that

$$a_n = \sum_{c \geq 0} a_n(c). \tag{1.1}$$

(The cyclomatic index is the number of *independent* cycles; it is the maximum number of edges that can be removed without disconnecting the animal.)

In the absence of a pleasing expression for a_n or for $a_n(c)$ there is some interest in determining the asymptotic behaviour for large n. Following Klarner (1967) it can be shown that

$$0 < \lim_{n\to\infty} n^{-1} \log a_n = \sup_{n>0} n^{-1} \log a_n \equiv \log \lambda < \infty \tag{1.2}$$

and Klein (1981) used similar concatenation arguments to show that

$$0 < \lim_{n\to\infty} n^{-1} \log a_n(0) = \sup_{n>0} n^{-1} \log a_n(0) \equiv \log \lambda_0 < \infty. \tag{1.3}$$

One can derive bounds on $a_n(c)$ in terms of $a_n(0)$, on the d-dimensional hypercubic lattice, and the best such bounds which have appeared are

$$a_n(c) \leq (2dn)^c a_n(0) \tag{1.4}$$

for $c \geq 1$ and all n (Whittington *et al* 1983), and

$$a_{n+c}(c) \geq A\binom{\varepsilon n}{c} a_n(0)/3^c \tag{1.5}$$

for some positive constant A and for all positive ε less than a positive constant $\varepsilon_0/5$, and n sufficiently large (Soteros and Whittington 1988). These bounds imply that

$$\lim_{n\to\infty} n^{-1} \log a_n(c) = \log \lambda_0 \tag{1.6}$$

for all c, where the limit is taken with c fixed. One can also use equations (1.4) and (1.5) to derive results on the critical exponents for c animals (Soteros and Whittington 1988).

Equation (1.6) is concerned with the number of c animals with c fixed, and so with the number of cycles per vertex equal to zero in the $n \to \infty$ limit. We can also consider the number of animals, $a_n(\alpha, \leq)$, with n vertices, having at most $\lfloor \alpha n \rfloor$ cycles. We use $\lfloor x \rfloor$ to mean the largest integer less than or equal to x and we use $\lceil x \rceil$ to mean the smallest integer greater than or equal to x. As $n \to \infty$ the maximum allowed number of cycles per vertex is α, which can be greater than zero. We shall show that

$$\lim_{n\to\infty} n^{-1} \log a_n(\alpha, \leq) = \log \lambda(\alpha) \tag{1.7}$$

exists, that $\lambda(\alpha)$ is a log concave function of α in $[0, d-1)$, and is continuous in this interval. In particular $\lim_{\alpha\to 0} \lambda(\alpha) = \lambda_0$.

The values of λ_0 and λ have been estimated numerically (Gaunt 1980, Gaunt *et al* 1982). These estimates are

$$\begin{aligned} \lambda_0 &= 5.14 \pm 0.01 \\ \lambda &= 5.210 \pm 0.006 \qquad (d=2) \\ \lambda_0 &= 10.53 \pm 0.07 \\ \lambda &= 10.62 \pm 0.08 \qquad (d=3). \end{aligned}$$

On the basis of these numerical results and an expansion in inverse powers of the dimension (d), Gaunt *et al* conjectured that

$$\lambda_0 < \lambda. \tag{1.8}$$

We give a rigorous proof of this result.

By improving some results on the numbers of trees (Soteros and Whittington 1988) we derive bounds on the α dependence of $\lambda(\alpha)$ which are sharp for small values of α. This is a regime which is difficult to investigate by standard numerical techniques. If $\phi(\alpha) = \lim_{n\to\infty} a_n(\lceil \alpha n \rceil)^{1/n}$ then

$$\phi(\alpha) = \lambda(\alpha) \tag{1.9}$$

for $\alpha \leq \alpha_0$, where

$$\alpha_0 = \sup\{\alpha | \lambda(\alpha) < \lambda(\infty)\}. \tag{1.10}$$

For $\alpha \leq \alpha_0$, $\lambda(\alpha)$ and $\phi(\alpha)$ yield equivalent information but, beyond α_0, the study of $\phi(\alpha)$ is more informative. We show that $\phi(\alpha)$ is log concave and continuous, that its derivative is infinite at the endpoints of the interval $[0, d-1)$, and that $\alpha_0 \leq (d-1)/2$.

Lubensky and Isaacson (1979) focused on the z transforms

$$A_n(z) = \sum_c a_n(c) z^c \tag{1.11}$$

where z plays the role of a cycle fugacity. We prove a corresponding set of results about the limit

$$\log \Lambda(z) = \lim_{n\to\infty} n^{-1} \log A_n(z) \tag{1.12}$$

and derive bounds on $\Lambda(z)$. We show that $\Lambda(z)$ and $\phi(\alpha)$ are connected through the Legendre transform

$$\log \Lambda(z) = \sup_{0\leqslant\alpha\leqslant d-1} (\log \phi(\alpha) + \alpha \log z). \tag{1.13}$$

2. General properties of $\lambda(\alpha)$

This section is concerned with investigating the functional form of $\lambda(\alpha)$, as defined in equation (1.7). In particular we prove that $\lim_{\alpha\to 0} \lambda(\alpha) = \lambda_0 < \lambda$.

On the d-dimensional hypercubic lattice a vertex has coordinates $(x_1, x_2, \ldots, x_d)$. We shall need two definitions. For any set S_0 of vertices we define the top (bottom) vertex as follows. First construct the subset $S_1 \subset S_0$ such that the coordinate x_1 of every vertex in S_1 has the maximum (minimum) value over all vertices in S_0. We then recursively construct $S_k \subset S_{k-1}$ such that the coordinate x_k of every vertex in S_k has the maximum (minimum) value over all vertices in S_{k-1}. Let j be the smallest integer such that S_j contains precisely one vertex, and call this vertex $t(b)$, the top (bottom) vertex of S_0.

Lemma 2.1. For an animal with n vertices on the d-dimensional hypercubic lattice the cyclomatic index c satisfies the inequality $c < n(d-1)$. For each n, define $c_{\max}(n) = \max\{c\colon a_n(c) > 0\}$; then $\lim_{n\to\infty} c_{\max}(n)/n = d-1$.

Proof. If a connected graph has n vertices and e edges the cyclomatic index c is given by (e.g. Berge 1962)

$$c = e - n + 1. \tag{2.1}$$

Since the coordination number of a d-dimensional hypercubic lattice is $2d$, the number of edges incident on a vertex is at most $2d$. In addition, every animal has a top and a bottom vertex and those vertices can have maximum degree $(2d-2)$. Each edge is incident on two vertices so that

$$2e \leqslant 2d(n-2) + 2(2d-2). \tag{2.2}$$

Equations (2.1) and (2.2) give

$$c \leqslant (d-1)n - 1. \tag{2.3}$$

Hence, $a_n(c) = 0$ if $c \geqslant n(d-1)$.

Since $c \leqslant n(d-1) - 1$, $c_{\max}(n)/n < d-1$. For a lower bound, let $m \equiv m(n) = \lfloor n^{1/d} \rfloor$. Since $n \geqslant m^d$, there exists an animal having n vertices including all of the vertices $\{(x_1, \ldots, x_d)\colon 1 \leqslant x_i \leqslant m, i = 1, \ldots, d\}$ as well as all of the edges connecting nearest neighbours. Then this animal has at least $(m-1)^d d$ edges, so it has at least $(m-1)^d d - n + 1$ cycles. Therefore $c_{\max}(n) \geqslant (n^{1/d} - 2)^d d - n + 1$. The lemma now follows.

Lemma 2.2. The limit $\lim_{n\to\infty} n^{-1} \log a_n(\alpha, \leqslant)$ exists and is a log concave function of α for $0 \leqslant \alpha < d-1$.

Proof. Consider an animal with n_1 vertices, and at most $\alpha_1 n_1$ cycles. This animal can be concatenated with an animal having n_2 vertices and at most $\alpha_2 n_2$ cycles by translating the animals so that the top vertex of the first animal is one lattice space in the x_1 direction below the bottom vertex of the second animal. If these two vertices are joined by an edge, the resulting animal has n_1+n_2 vertices and at most $\alpha_1 n_1+\alpha_2 n_2$ cycles. Since each pair of animals can be concatenated in this way to form a distinct animal, we have

$$a_{n_1}(\alpha_1, \leqslant)a_{n_2}(\alpha_2, \leqslant) \leqslant a_{n_1+n_2}[(\alpha_1 n_1+\alpha_2 n_2)/(n_1+n_2), \leqslant]. \tag{2.4}$$

If we set $\alpha_1=\alpha_2=\alpha$ this gives

$$a_{n_1}(\alpha, \leqslant)a_{n_2}(\alpha, \leqslant) \leqslant a_{n_1+n_2}(\alpha, \leqslant). \tag{2.5}$$

This, together with the fact that

$$a_n(\alpha, \leqslant) \leqslant a_n \tag{2.6}$$

and $a_n^{1/n}$ is bounded above (as in Klarner 1967), gives the existence of the limit

$$\lim_{n\to\infty} n^{-1} \log a_n(\alpha, \leqslant) = \log \lambda(\alpha). \tag{2.7}$$

Now put $n_1=n_2=n$ in equation (2.4). This gives

$$a_n(\alpha_1, \leqslant)a_n(\alpha_2, \leqslant) \leqslant a_{2n}[(\alpha_1+\alpha_2)/2, \leqslant]. \tag{2.8}$$

Taking logarithms, dividing by n and letting $n\to\infty$, gives

$$\log \lambda(\alpha_1)+\log \lambda(\alpha_2) \leqslant 2 \log \lambda[(\alpha_1+\alpha_2)/2]. \tag{2.9}$$

Since $\lambda(\alpha)$ is a bounded non-decreasing function of α then equation (2.9) implies that $\lambda(\alpha)$ is a log concave function of α for $\alpha \in [0, d-1)$ (Hardy *et al* 1934, § 3.18).

Lemma 2.3.

$$\binom{c+k}{k} a_n(c+k) \leqslant \binom{(d-1)n-c-1}{k} a_n(c) \tag{2.10}$$

and therefore

$$c^k a_n(c+k) \leqslant [(d-1)n]^k a_n(c). \tag{2.11}$$

Also, if $(d-1)/2 \leqslant c_1 \leqslant c_2 \leqslant c_{\max}(n)$, then

$$a_n(c_1) \geqslant a_n(c_2). \tag{2.12}$$

Proof. Let $b_n(c, c+k)$ be the number of pairs of animals $\{g_n(c), g_n(c+k)\}$ such that $g_n(c)$ has n vertices and c cycles, $g_n(c+k)$ has *the same* n vertices and $c+k$ cycles and includes all the edges of $g_n(c)$. An animal with $c+k$ cycles has at least one set of $c+k$ edges which can be simultaneously deleted to yield a connected graph. We pick one such set of $c+k$ edges and we can then choose to remove k of these in $\binom{c+k}{k}$ ways, yielding an animal with n vertices and c cycles, so that

$$b_n(c, c+k) \geqslant \binom{c+k}{k} a_n(c+k). \tag{2.13}$$

For each $g_n(c)$ we now seek an upper bound on the number of associated graphs $g_n(c+k)$. Since any animal with n vertices can have at most $n(d-1)-1$ cycles (lemma 2.1) there are at most $[n(d-1)-1]-c$ edges which can be added, and we choose to add k of these to form k additional cycles. These k edges can be chosen in at most $\binom{(d-1)n-c-1}{k}$ ways so that

$$b_n(c, c+k) \leqslant \binom{(d-1)n-c-1}{k} a_n(c). \tag{2.14}$$

Then equations (2.13) and (2.14) imply equation (2.10). Equations (2.11) and (2.12) follow easily from equation (2.10).

Lemma 2.4.

$$\lim_{n\to\infty} n^{-1} \log \binom{an}{bn} = a \log a - b \log b - (a-b) \log(a-b). \tag{2.15}$$

Proof. The bound

$$n \log n - n < \log n! < (n+1) \log(n+1) - n \tag{2.16}$$

(see Feller 1950) is sufficient to give the result in (2.15).

Theorem 1. $\log \lambda(\alpha)$ is continuous for $0 \leqslant \alpha < d-1$.

Proof. Since $\log \lambda(\alpha)$ is a non-decreasing concave function of α in $[0, d-1)$ it is continuous in $(0, d-1)$. Hence we need only establish continuity at $\alpha = 0$. We do this using the upper bound (2.10), with $c = 0$. We have

$$\begin{aligned} a_n(\alpha, \leqslant) &= \sum_{k=0}^{\lfloor \alpha n \rfloor} a_n(k) \\ &\leqslant \sum_{k=0}^{\lfloor \alpha n \rfloor} \binom{(d-1)n-1}{k} a_n(0) \\ &\leqslant (\lfloor \alpha n \rfloor + 1) a_n(0) \max_{k \leqslant \lfloor \alpha n \rfloor} \binom{(d-1)n-1}{k} \\ &= (\lfloor \alpha n \rfloor + 1) a_n(0) \binom{(d-1)n-1}{\lfloor \alpha n \rfloor} \end{aligned} \tag{2.17}$$

provided that $\lfloor \alpha n \rfloor \leqslant [(d-1)n-1]/2$. So for α small enough

$$\log \lambda(\alpha) = \lim_{n\to\infty} n^{-1} \log a_n(\alpha, \leqslant) \leqslant \log \lambda_0 + \lim_{n\to\infty} n^{-1} \log \binom{(d-1)n-1}{\lfloor \alpha n \rfloor} \tag{2.18}$$

and by lemma 2.4

$$\log \lambda(\alpha) \leqslant \log \lambda_0 + (d-1) \log(d-1) - (\alpha) \log(\alpha) - [(d-1)-\alpha] \log[(d-1)-\alpha]. \tag{2.19}$$

Then letting $\alpha \to 0+$ gives

$$\lim_{\alpha\to 0+} \log \lambda(\alpha) = \log \lambda_0. \tag{2.20}$$

Theorem 2. $\lambda > \lambda_0$.

Proof. For this proof we obtain a lower bound on $\log \lambda(\alpha)$ using the lower bound in equation (1.5). We have, for n sufficiently large,

$$\begin{aligned} a_n(\alpha, \leqslant) &= \sum_{k=0}^{\lfloor \alpha n \rfloor} a_n(k) \\ &\geqslant \sum_{k=0}^{\lfloor \alpha n \rfloor} A\binom{\varepsilon(n-k)}{k} a_{n-k}(0) 3^{-k} \\ &\geqslant A\binom{\varepsilon(n-\lfloor \alpha n \rfloor)}{\lfloor \alpha n \rfloor} a_{n-\lfloor \alpha n \rfloor}(0) 3^{-\lfloor \alpha n \rfloor} \end{aligned} \tag{2.21}$$

provided that $\alpha \leqslant [\varepsilon/(1+\varepsilon)]$ and $\varepsilon < \varepsilon_0/5$. Hence

$$\begin{aligned} \log \lambda(\alpha) &= \lim_{n\to\infty} n^{-1} \log a_n(\alpha, \leqslant) \\ &\geqslant \varepsilon(1-\alpha) \log \varepsilon(1-\alpha) - \alpha \log \alpha \\ &\quad -[\varepsilon(1-\alpha)-\alpha] \log[\varepsilon(1-\alpha)-\alpha] + (1-\alpha) \log \lambda_0 - \alpha \log 3 \end{aligned} \tag{2.22}$$

and thus

$$\log \lambda(\alpha) - \log \lambda_0 \geqslant \varepsilon(1-\alpha) \log \frac{[\varepsilon(1-\alpha)]}{[\varepsilon(1-\alpha)-\alpha]} + \alpha \log \frac{(\varepsilon - \varepsilon\alpha - \alpha)}{\alpha 3 \lambda_0}. \tag{2.23}$$

The first term on the right-hand side of equation (2.23) is always positive and the second is positive for α such that $0 < \alpha < \varepsilon/(1+3\lambda_0+\varepsilon)$. Hence there exists $\alpha > 0$ such that $\log \lambda(\alpha) > \log \lambda_0$. Since $\lambda(\alpha)$ is non-decreasing and bounded above by λ, $\lambda > \lambda_0$ and theorem 2 is proved.

3. A lower bound on $\lambda(\alpha)$

Equation (2.19) gives an upper bound on $\lambda(\alpha)$ when α is small. In this section we derive a corresponding lower bound on $\lambda(\alpha)$. The strategy is to make use of (2.23) by finding a lower bound on ε_0. This involves sharpening an upper bound on the number of trees derived by Soteros and Whittington (1988) (their equation (2.10)). We define $t_n(\alpha, \leqslant)$ to be the number of trees with n vertices containing at most αn vertices of degree greater than 2, and

$$\log \lambda_0(\alpha) = \lim_{n\to\infty} n^{-1} \log t_n(\alpha, \leqslant) \tag{3.1}$$

(where the limit is known to exist (Lipson and Whittington 1983)). The expected behaviour (using the results of Soteros and Whittington (1988)) of $\lambda_0(\alpha)$ is shown in figure 1. ε_0 is the smallest value of α at which $\lambda_0(\alpha) = \lambda_0(1) \equiv \lambda_0$. We derive an upper bound on $\lambda_0(\alpha)$, sketched as the long broken curve in figure 1, and establish the point at which this bound meets a lower bound on λ_0, derived by Whittington and Gaunt (1978), the horizontal broken line. The value of α at which these bounds meet, $\alpha = \varepsilon^*$, is a lower bound on ε_0.

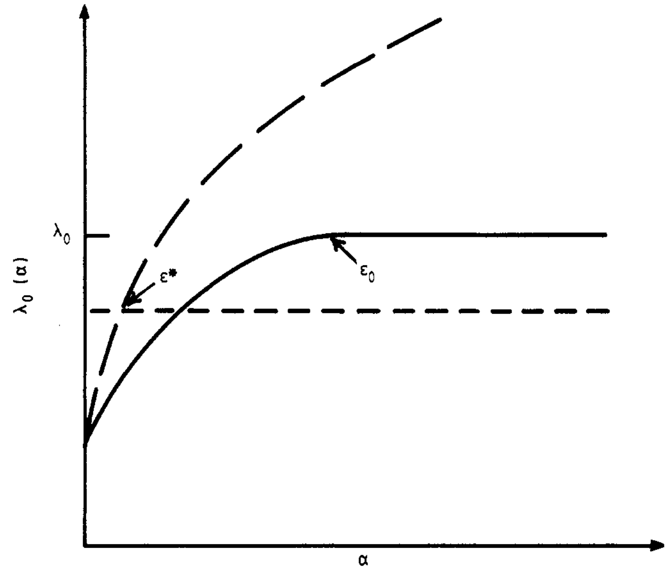

Figure 1. Expected behaviour of $\lambda_0(\alpha)$. The long broken curve represents an upper bound on $\lambda_0(\alpha)$ and the horizontal broken line represents a lower bound on the value of λ_0. ε^* is then a lower bound on ε_0.

Let $u_n(\alpha)$ be the number of trees with n vertices having at most αn vertices of degree not equal to 2. For a tree on a d-dimensional hypercubic lattice we can define n_i, with $1 \leq i \leq 2d$, to be the number of vertices having degree i. Hence for $0 \leq \alpha \leq [1/(2d-1)]-[2/n(2d-1)]$

$$t_n(\alpha, \leq) \leq u_n[(2d-1)\alpha + 2/n] \tag{3.2}$$

since

$$n_1 + \sum_{i=3}^{2d} n_i = 2 + \sum_{i=3}^{2d} (i-1)n_i \leq 2 + (2d-1)\alpha n. \tag{3.3}$$

Using equations (3.1) and (3.2) an upper bound on $u_n[(2d-1)\alpha + 2/n]$ will lead to an upper bound on $\lambda_0(\alpha)$.

To obtain an upper bound on $u_n[(2d-1)\alpha + 2/n]$ we need to define an ordering of the branches and branch points of a tree, T, having b vertices of degree not equal to 2 on the d-dimensional hypercubic lattice. We do this by considering the abstract homeomorphically irreducible tree, τ, having b vertices, which is associated with the lattice tree, T. By abstract we mean that τ is a tree, in the graph theoretic sense, which is not embedded in any space. For τ we define the following ordering of its branches and branch points (there are $b - n_1$ branch points). First note that the vertices of the tree can be ranked according to degree (and, in case of ambiguity, sums of degrees of adjacent vertices, see Bersohn (1978)). Using this ranking, we choose a vertex of highest rank (there can be equivalently ranked vertices using this ranking) to be the first vertex. The second vertex is chosen to be a vertex having highest rank among those vertices adjacent to the first vertex. The first branch is the branch connecting the first vertex to the second vertex. The remaining vertices adjacent to the first vertex are ordered sequentially according to their ranking. The branches connecting these vertices to the first vertex are also ordered sequentially following the same order. The ordering process continues from the vertices adjacent to the second vertex and the process ends when all $b-n_1$ branch points and $b-1$ branches are ordered. There is more than one such ordering of the branches and branch points of τ so we fix one ordering for each τ. Thus for any lattice tree T we use the ordering of the branches and branch points of the associated abstract homeomorphically irreducible tree τ to order the branches and branch points of T. In particular we can talk about the number of steps in each of the $b-1$ branches of T as $m_1, m_2, \ldots, m_{b-1}$, respectively.

Let $t_n^\tau(m_1, m_2, \ldots, m_{b-1})$ be the number of n-vertex lattice trees T having $b-1$ branches containing $m_1, m_2, \ldots, m_{b-1}$ steps respectively, such that T is associated with the abstract homeomorphically irreducible tree on b vertices, τ. Then we have the following lemma.

Lemma 3.1.

$$t_n^\tau(m_1, m_2, \ldots, m_{b-1}) \leqslant 2d \left[\prod_{i=3}^{2d} \left(\frac{(2d-1)!}{(2d-i)!}\right)^{n_i}\right]\left(\prod_{k=1}^{b-1} f(m_k)/2d\right) \tag{3.4}$$

where $f(m_k)$ is the number of self-avoiding walks with m_k steps on the d-dimensional hypercubic lattice and n_i is the number of vertices of degree i in the tree τ.

Proof. We can bound $t_n^\tau(m_1, m_2, \ldots, m_{b-1})$ above by the number of ways of embedding branches independently in the lattice, in the order defined for τ, so that the branches have the correct lengths $m_1, m_2, \ldots, m_{b-1}$. The number of ways of embedding the first branch is bounded above by $f(m_1)$. For the remaining $(i-1)$ branches (i is the degree of the branch point) connected to the first branch there are $(2d-1)!/(2d-i)!$ ways to choose their first steps. The number of ways to embed any one of these branches, where the first step is fixed, is bounded above by $f(m_k)/2d$, where m_k is the number of steps in the branch. At the next (according to the order defined for τ) branch point a similar bound results since again one of its branches has already been embedded and there are $(2d-1)!/(2d-i)!$ choices for the first steps of the remaining $(i-1)$ branches. Multiplying together the bounds calculated in this manner at each branch point gives the bound in equation (3.4).

Lemma 3.2.

$$u_n(\beta) \leqslant \sum_{b=2}^{\beta n} 2dB\theta^{-b}\left[\prod_{i=3}^{2d}\left(\frac{(2d-1)!}{(2d-i)!}\right)^{n_i}\right]\sum_{j=1}^{\binom{n-2}{b-2}}\left(\prod_{k=1}^{b-1} f(m_{jk})/2d\right) \tag{3.5}$$

for constants $B>0$ and $\theta<1$. The summation over j is a sum over the number of ways of distributing $n-1$ edges amongst the $b-1$ branches of the tree with at least one edge per branch, $m_{j1}, m_{j2}, \ldots, m_{j(b-1)}$ are the number of edges in each branch respectively for the jth way of distributing the edges.

Proof.

$$u_n(\beta) = \sum_{b=2}^{\beta n}\sum_{\tau}\sum_{m_1, m_2, \ldots, m_{b-1}} t_n^\tau(m_1, m_2, \ldots, m_{b-1}) \tag{3.6}$$

where the summation over τ is a sum over all abstract homeomorphically irreducible trees on b vertices. The third sum is over all possible choices of $m_1, m_2, \ldots, m_{b-1}$, the lengths of the branches.

Since the right-hand side of equation (3.4) is independent of τ we can use equation (3.4) and equation (3.6) to give

$$u_n(\beta) \leq \sum_{b=2}^{\beta n} H(b) \sum_{m_1, m_2, \ldots, m_{b-1}} 2d \left[\prod_{i=3}^{2d} \left(\frac{(2d-1)!}{(2d-i)!} \right)^{n_i} \right] \left(\prod_{k=1}^{b-1} f(m_k)/2d \right) \tag{3.7}$$

where $H(b)$ is the number of abstract (unlabelled) homeomorphically irreducible trees on b vertices.

Since there are $\binom{n-2}{b-2}$ ways to distribute the $n-1$ edges amongst the $b-1$ branches of the tree with at least one edge in every branch and hence $\binom{n-2}{b-2}$ choices for $m_1, m_2, \ldots, m_{b-1}$ we can rewrite the right-hand side of equation (3.7) to give

$$u_n(\beta) \leq \sum_{b=2}^{\beta n} 2dH(b) \left[\prod_{i=3}^{2d} \left(\frac{(2d-1)!}{(2d-i)!} \right)^{n_i} \right] \sum_{j=1}^{\binom{n-2}{b-2}} \left(\prod_{k=1}^{b-1} f(m_{jk})/2d \right) \tag{3.8}$$

where $m_{j1}, m_{j2}, \ldots, m_{j(b-1)}$ are the number of edges in each branch respectively for the jth way of distributing the edges.

From Harary *et al* (1975) and Harary and Prins (1959) it can be concluded that there exist constants $B > 0$ and $\theta < 1$ such that $H(b) \leq B\theta^{-b}$ where $\theta = 0.456\,733\ldots$. This, along with equation (3.8), gives equation (3.5).

The functional form of $f(n)$ is not known but equation (3.5) can be simplified by replacing $f(n)$ by an upper bound. We obtain a simpler form for equation (3.5) using the following lemmas.

Lemma 3.3. Suppose that $g(n)$ is a twice differentiable function and $g''(n) < 0$ for all n. If $f(n) \leq \mathrm{e}^{g(n)}$ for $n \in Z$, then

$$u_n(\beta) \leq \sum_{b=2}^{\beta n} \frac{B}{\theta^b (2d)^{b-2}} \binom{n-2}{b-2} \left(\frac{(2d-1)!}{(2d-3)!} \right)^{(b-2)/2} [\![\exp\{g[(n-1)/(b-1)]\}]\!]^{b-1}. \tag{3.9}$$

Proof. $g(n)$ is a function of n with $g''(n) < 0$ and therefore by convexity

$$\max_{\{m_i | \Sigma_{i=1}^{b-1} m_i = n-1, m_i \geq 1\}} \sum_{i=1}^{b-1} g(m_i) = (b-1) g\left(\frac{n-1}{b-1} \right). \tag{3.10}$$

Since $\Pi_{j=3}^{i-1}(2d-j) < [(2d-1)(2d-2)]^{(i-3)/2}$

$$\max_{\{n_i | \Sigma_{i=3}^{2d}(i-1)n_i = b-2\}} \prod_{i=3}^{2d} \left(\frac{(2d-1)!}{(2d-i)!} \right)^{n_i} = \left(\frac{(2d-1)!}{(2d-3)!} \right)^{(b-2)/2}. \tag{3.11}$$

Equations (3.5), (3.10) and (3.11) together with $f(n) \leq \mathrm{e}^{g(n)}$ give equation (3.9).

We now use lemma 3.3 and an appropriate upper bound on $f(n)$ to obtain an upper bound on $u_n(\beta)$. This upper bound along with equation (3.2), where $\beta = (2d-1)\alpha + 2/n$, gives an upper bound on $t_n(\alpha, \leq)$ and then taking logarithms, dividing by n and letting n go to infinity gives the following lemma.

Lemma 3.4. For

$$\alpha \leqslant \left(\frac{q}{1+q}\right)\frac{1}{2d-1}$$

where

$$q=(\mathrm{e}^{\kappa}/2d\theta)[(2d-1)(2d-2)]^{1/2}$$

we have

$$\begin{aligned}\log \lambda_0(\alpha) \leqslant \kappa + \pi\{(2d-1)\alpha[1+(2d-1)\alpha]\}^{1/2} &+ (2d-1)\alpha \log[1+(2d-1)\alpha] \\ &-2(2d-1)\alpha \log(2d-1)\alpha \\ &-[1-(2d-1)\alpha]\log[1-(2d-1)\alpha] \\ &+(2d-1)\alpha \log\left(\frac{\mathrm{e}^{\kappa}}{2d\theta}[(2d-1)(2d-2)]^{1/2}\right) \end{aligned} \tag{3.12}$$

while for

$$\left(\frac{q}{1+q}\right)\frac{1}{2d-1} < \alpha \qquad \alpha^* = \min\{\alpha, 1/(2d-1)\}$$

we have

$$\begin{aligned}\log \lambda_0(\alpha) \leqslant \kappa + \pi\{(2d-1)\alpha^*[1+(2d-1)\alpha^*]\}^{1/2} &+ (2d-1)\alpha^* \log[1+(2d-1)\alpha^*] \\ -(2d-1)\alpha^* \log[(2d-1)\alpha^*] &+ \log\left(1+\frac{\mathrm{e}^{\kappa}}{2d\theta}[(2d-1)(2d-2)]^{1/2}\right)\end{aligned} \tag{3.13}$$

where κ is the connective constant for self-avoiding walks.

Proof. Since an exact equation for $f(n)$ is not known we use the upper bound on $f(n)$ derived by Hammersley and Welsh (1962). We use their bound in the following form:

$$f(n) \leqslant \exp[(n+1)\kappa] \sum_{r=0}^{n} p_{\mathrm{D}}(r) p_{\mathrm{D}}(n-r+1) \tag{3.14}$$

where $p_{\mathrm{D}}(r)$ is the number of partitions of r into distinct integers or, equivalently, the number of partitions of r into an odd number of parts. Following Hua's (1982) proof that the number of partitions of n is less than $\exp[\pi(3n/2)^{1/2}]$, it can be shown that

$$p_{\mathrm{D}}(r) < \exp[\pi(r/2)^{1/2}] \tag{3.15}$$

for all finite r. Therefore we have from equation (3.14) that

$$f(n) \leqslant \exp[(n+1)\kappa] \sum_{r=0}^{n} \exp\{(\pi/\sqrt{2})[r^{1/2}+(n-r+1)^{1/2}]\}. \tag{3.16}$$

The sum over r in equation (3.16) attains its maximum for $r=(n+1)/2$ and therefore

$$f(n) \leqslant \exp[(n+1)\kappa](n+1)\exp[\pi(n+1)^{1/2}]. \tag{3.17}$$

We let the logarithm of the right-hand side of equation (3.17) be $g(n)$ in lemma 3.3, i.e.

$$g(n)=(n+1)\kappa+\pi(n+1)^{1/2}+\log(n+1). \tag{3.18}$$

We therefore have that

$$u_n(\beta)\leqslant\sum_{b=2}^{\beta n}\frac{B}{\theta^b(2d)^{b-2}}\binom{n-2}{b-2}\left(\frac{(2d-1)!}{(2d-3)!}\right)^{(b-2)/2}$$
$$\times\left\{\exp\left[\left(\frac{n+b-2}{b-1}\right)\kappa+\pi\left(\frac{n+b-2}{b-1}\right)^{1/2}+\log\left(\frac{n-1}{b-1}+1\right)\right]\right\}^{b-1}. \tag{3.19}$$

Note that the functions $[(n-1)/(b-1)+1]^{b-1}$ and $\exp\{\pi[(b-1)(n+b-2)]^{1/2}\}$ are both increasing functions of b for $b\leqslant n$ and hence we can replace them by their value at the maximum b value and obtain

$$u_n(\beta)\leqslant(B/\theta^2)\exp\{n\kappa+\pi[(n+\beta n-2)(\beta n-1)]^{1/2}$$
$$+(\beta n-1)\log[(n+\beta n-2)/(\beta n-1)]\}$$
$$\times\sum_{b=2}^{\beta n}\binom{n-2}{b-2}\left(\frac{e^\kappa}{2d\theta}\right)^{b-2}\left(\frac{(2d-1)!}{(2d-3)!}\right)^{(b-2)/2}. \tag{3.20}$$

We consider two different ranges of β and in each case bound the right-hand side of equation (3.20). These two ranges are for $\beta\in[0,q/(1+q))$ and $\beta\in[q/(1+q),1]$ where $q=(e^\kappa/2d\theta)[(2d-1)(2d-2)]^{1/2}$.

For the first case, $0\leqslant\beta\leqslant q/(1+q)$, it can be proved that for any q such that $0<q<\infty$

$$\sum_{k=0}^{\beta n}\binom{n}{k}q^k\leqslant\left(\frac{q^\beta}{\beta^\beta(1-\beta)^{1-\beta}}\right)^n. \tag{3.21}$$

To prove this, define $p=q/(1+q)$. Thus we have that $0<\beta<p<1$ and the left-hand side of equation (3.21) can be rewritten as $\Sigma_{k=0}^{\beta n}\binom{n}{k}p^k(1-p)^{n-k}$. Let X be a binomial random variable with parameters n and p. Then $\Sigma_{k=0}^{\beta n}\binom{n}{k}p^k(1-p)^{n-k}=\Pr(X\leqslant\beta n)$. Define the random variable Z which is 1 if $X\leqslant\beta n$ and 0 otherwise. Then $Z\leqslant\exp[t(\beta n-X)]$ for any $t\geqslant0$, so that

$$\begin{aligned}\Pr\{X\leqslant\beta n\}&=E(Z)\\&\leqslant E(\exp[t(\beta n-X)])\\&=e^{t\beta n}E(e^{-tX})\\&=e^{t\beta n}(p\,e^{-t}+1-p)^n.\end{aligned} \tag{3.22}$$

The last expression in equation (3.22) is minimised (over t) when $\beta(1-p)=p(1-\beta)e^{-t}$; for this value of t, we obtain

$$\sum_{k=0}^{\beta n}\binom{n}{k}p^k(1-p)^{n-k}\leqslant\left[\left(\frac{p}{\beta}\right)^\beta\left(\frac{1-p}{1-\beta}\right)^{1-\beta}\right]^n. \tag{3.23}$$

(Note that if we take nth roots and let $n\to\infty$ in equation (3.23), we obtain equality (Cramér 1937).) Substituting $p=q/(1+q)$ into equation (3.23) gives equation (3.21).

Substituting equation (3.21) into equation (3.20) gives for $0\leqslant\beta\leqslant q/(1+q)$:

$$u_n(\beta)\leqslant\frac{B}{\theta^2}\frac{q^{\beta n}}{\beta^{\beta n}(1-\beta)^{(1-\beta)n}}\exp\{n\kappa+\pi[(n+\beta n-2)(\beta n-1)]^{1/2}$$
$$+(\beta n-1)\log[(n+\beta n-2)/(\beta n-1)]\}. \tag{3.24}$$

For the second case, note that the function summed over b in equation (3.20) takes its maximum for $b=[q/(1+q)]n$ and therefore for $q/(1+q)\leqslant\beta\leqslant 1$ using equation (3.20) we obtain

$$u_n(\beta)\leqslant\frac{B}{\theta^2}(\beta n-2)\binom{n-2}{\frac{q}{1+q}n-2}q^{[q/(1+q)]n-2}$$
$$\times\exp\{n\kappa+\pi[(n+\beta n-2)(\beta n-1)]^{1/2}$$
$$+(\beta n-1)\log[(n+\beta n-2)/(\beta n-1)]\}. \tag{3.25}$$

Taking logarithms, dividing by n and letting $n\to\infty$ in equations (3.24) and (3.25) gives for cases 1 and 2 respectively:

$$\lim_{n\to\infty}(1/n)\log u_n(\beta)$$
$$\leqslant\kappa+\pi[(1+\beta)\beta]^{1/2}+\beta\log(1+\beta)-2\beta\log\beta$$
$$+(1-\beta)\log(1-\beta)$$
$$+\beta\log\{(e^{\kappa}/2d\theta)[(2d-1)(2d-2)]^{1/2}\} \tag{3.26}$$

$$\lim_{n\to\infty}(1/n)\log u_n(\beta)$$
$$\leqslant\kappa+\pi[(1+\beta)\beta]^{1/2}+\beta\log(1+\beta)-\beta\log\beta$$
$$+\log\{1+(e^{\kappa}/2d\theta)[(2d-1)(2d-2)]^{1/2}\}. \tag{3.27}$$

For $\alpha\geqslant 1/(2d-1)$, $t_n(\alpha,\leqslant)\leqslant u_n(1)$ and for $\alpha<1/(2d-1)$, as in equation (3.2), $t_n(\alpha,\leqslant)\leqslant u_n[(2d-1)\alpha+2/n]$. Therefore taking logarithms, dividing by n and letting $n\to\infty$, we have that for $\alpha\geqslant 1/(2d-1)$, $\log\lambda_0(\alpha)\leqslant\lim_{n\to\infty}(1/n)\log u_n(1)$ and for $\alpha<1/(2d-1)$, $\log\lambda_0(\alpha)\leqslant\lim_{n\to\infty}(1/n)\log u_n[(2d-1)\alpha]$. For $\alpha\leqslant[q/(1+q)][1/(2d-1)]$ equation (3.26) with $\beta=(2d-1)\alpha$ gives equation (3.12), for $[q/(1+q)][1/(2d-1)]<\alpha<1/(2d-1)$ equation (3.27) with $\beta=(2d-1)\alpha$ gives equation (3.13) where $\alpha^*=\alpha$ and for $\alpha\geqslant 1/(2d-1)$ equation (3.27) with $\beta=1$ gives equation (3.13) where $\alpha^*=1/(2d-1)$.

An upper bound for κ on the two-dimensional square lattice is available from Fisher and Sykes (1959):

$$e^{\kappa}<2.712. \tag{3.28}$$

A lower bound on λ_0 for the two-dimensional square lattice is available from Whittington and Gaunt (1978):

$$\lambda_0\geqslant 4.3486. \tag{3.29}$$

The bound on κ together with equation (3.13) allows us to determine a lower bound on the first value of α for which $\lambda_0(\alpha)=\lambda_0$. This value of α is bounded below by the first value of α for which our upper bound on $\lambda_0(\alpha)$ equals the lower bound on λ_0 in equation (3.29). This value of α is a lower bound on ε_0 needed in equation (2.22). The result for the two-dimensional lattice is that $\varepsilon>0.003\,899$. If, instead of using the bounds on λ_0 and κ, as in equations (3.28) and (3.25), we use the series estimates $\lambda_0\approx 5.14$ (Gaunt *et al* 1982) and $e^{\kappa}\approx 2.6381$ (Sykes *et al* 1972) then we obtain $\varepsilon_0>0.006\,958$. Equation (2.19) and equation (2.23) with ε replaced by $(0.003\,899)/5$ give upper and lower bounds on $\lambda(\alpha)$ for small α.

4. Properties of $\phi(\alpha)$

In this section we shall investigate the function

$$\phi(\alpha) = \lim_{n\to\infty} a_n(\lceil \alpha n \rceil)^{1/n} \tag{4.1}$$

beginning with the existence of the limit. Some fundamental properties of ϕ including log concavity and continuity will then be proved in a series of lemmas.

Lemma 4.1. For $0 \leqslant \alpha < d-1$, $\lim_{n\to\infty} a_n(\lceil \alpha n \rceil)^{1/n}$ exists.

Proof. The result is true for $\alpha = 0$ by equation (1.3) and we therefore only consider $0 < \alpha < d-1$. The usual concatenation argument (cf lemma 2.2) implies

$$a_n(\lceil \alpha n \rceil) a_m(\lceil \alpha m \rceil) \leqslant a_{n+m}(\lceil \alpha n \rceil + \lceil \alpha m \rceil). \tag{4.2}$$

There are two possibilities.

(a) $\lceil \alpha n \rceil + \lceil \alpha m \rceil = \lceil \alpha(n+m) \rceil$. Then $a_n(\lceil \alpha n \rceil) a_m(\lceil \alpha m \rceil) \leqslant a_{n+m}(\lceil \alpha(n+m) \rceil)$.

(b) $\lceil \alpha n \rceil + \lceil \alpha m \rceil = \lceil \alpha(n+m) \rceil + 1$. By equation (2.11) of lemma 2.3 with $k=1$, we know

$$a_{n+m}(\lceil \alpha(n+m) \rceil + 1) \leqslant a_{n+m}(\lceil \alpha(n+m) \rceil) \frac{(d-1)(n+m)}{\lceil \alpha(n+m) \rceil} \tag{4.3}$$

so in both (a) and (b) we find

$$a_n(\lceil \alpha n \rceil) a_m(\lceil \alpha m \rceil) \leqslant \frac{d-1}{\alpha} a_{n+m}(\lceil \alpha(n+m) \rceil). \tag{4.4}$$

Thus, submultiplicativity, together with the fact that $a_n(\lceil \alpha n \rceil)^{1/n} \leqslant \lambda$, implies that

$$\lim_{n\to\infty} \left(\frac{\alpha}{d-1} a_n(\lceil \alpha n \rceil) \right)^{1/n}$$

exists. The lemma follows.

Now that lemma 4.1 is proved, we shall take (4.1) as the *definition* of $\phi(\alpha)$ for $0 \leqslant \alpha < d-1$; we also define $\phi(d-1) = 1$.

Lemma 4.2. For $0 < \varepsilon < d-1$

$$\phi(d-1-\varepsilon) \leqslant \frac{(d+\varepsilon)^{d+\varepsilon}}{(d-\varepsilon)^{d-\varepsilon}(2\varepsilon)^{2\varepsilon}}. \tag{4.5}$$

This implies

$$\lim_{\alpha\to(d-1)^-} \phi(\alpha) = 1. \tag{4.6}$$

Proof. Consider an animal with $\lceil n(d-1-\varepsilon) \rceil$ cycles and n vertices. Then $e \equiv \lceil nd - n\varepsilon - 1 \rceil$ is the number of edges, and by counting edge-vertex incidence pairs one sees that there are at most $2nd - 2e = 2\lceil n\varepsilon \rceil + 2$ 'boundary edges' (edges which are not in the animal but have at least one endpoint in the animal). Let $a_{n,e,i}$ be the number of animals with n vertices, e edges and i boundary edges. Then by arguing as in Kesten (1982, lemma 5.1) for the edge-percolation model, we find

$$1 \geqslant \sum_{i=1}^{\infty} n a_{n,e,i} p^{e}(1-p)^{i}$$
$$\geqslant \sum_{i=1}^{2\lceil n\varepsilon\rceil+2} n a_{n,e,i} p^{n(d-\varepsilon)}(1-p)^{2n\varepsilon+4}. \qquad (4.7)$$

Since $a_n(\lceil n(d-1-\varepsilon)\rceil) = \sum_{i=1}^{2\lceil n\varepsilon\rceil+2} a_{n,e,i}$, taking nth roots in the above inequality and using (4.1), we obtain

$$\phi(d-1-\varepsilon) \leqslant [p^{d-\varepsilon}(1-p)^{2\varepsilon}]^{-1} \qquad 0<p<1. \qquad (4.8)$$

The right-hand side is minimised when $p=(d-\varepsilon)/(d+\varepsilon)$, which gives (4.5).

Lemma 4.3.

$$\phi(\alpha) I\left(\frac{\alpha}{d-1}\right)^{d-1} \geqslant \phi(\beta) I\left(\frac{\beta}{d-1}\right)^{d-1} \qquad (4.9)$$

for $0 \leqslant \alpha \leqslant \beta \leqslant d-1$, where $I(t)=t^{t}(1-t)^{1-t}, 0<t<1$ and $I(0)=I(1)=1$. That is, $\phi(\alpha)[I(\alpha/(d-1))]^{d-1}$ is monotone decreasing for $0 \leqslant \alpha \leqslant d-1$.

Proof. Let

$$\Phi(\alpha)=\phi(\alpha)\left[I\left(\frac{\alpha}{d-1}\right)\right]^{d-1}.$$

For $0 \leqslant \alpha < \beta < d-1$, put $c=\lceil \alpha n\rceil$ and $k=\lceil \beta n\rceil - c$ in equation (2.10); then take nth roots and let $n \to \infty$. Using lemma 2.4 we obtain $\Phi(\alpha) \geqslant \Phi(\beta)$, so $\Phi(\alpha)$ is monotone decreasing for $0 \leqslant \alpha < d-1$. By (4.6), $\lim_{\alpha \to (d-1)-} \Phi(\alpha)=1$, and $\Phi(d-1)=1$, so the result follows.

The monotonicity of Φ and the continuity of I imply that all one-sided limits of ϕ exist, and that

$$\phi(\alpha-) \geqslant \phi(\alpha) \geqslant \phi(\alpha+). \qquad (4.10)$$

Lemma 4.4. ϕ is log concave and continuous on $[0, d-1]$.

Proof. For $\alpha, \beta \in [0, d-1)$

$$a_n(\lceil \alpha n\rceil) a_n(\lceil \beta n\rceil) \leqslant a_{2n}(\lceil \alpha n\rceil + \lceil \beta n\rceil) \qquad (4.11)$$

by the usual concatenation argument. There are two possibilities.

(a) $\lceil \alpha n\rceil + \lceil \beta n\rceil = \lceil(\alpha+\beta)n\rceil$.

(b) $\lceil \alpha n\rceil + \lceil \beta n\rceil = \lceil(\alpha+\beta)n\rceil + 1$, in which case (by equation (2.11))

$$a_{2n}(\lceil \alpha n\rceil + \lceil \beta n\rceil) \leqslant a_{2n}(\lceil(\alpha+\beta)n\rceil) \frac{(d-1)2n}{\lceil(\alpha+\beta)n\rceil}. \qquad (4.12)$$

In both cases, we find

$$a_n(\lceil \alpha n\rceil) a_n(\lceil \beta n\rceil) \leqslant a_{2n}(\lceil \tfrac{1}{2}(\alpha+\beta)2n\rceil) \frac{(d-1)2}{(\alpha+\beta)}. \qquad (4.13)$$

Take nth roots and let $n \to \infty$; this gives

$$\phi(\alpha)\phi(\beta) \leqslant \phi\left(\frac{\alpha+\beta}{2}\right)^2. \tag{4.14}$$

$\phi(\alpha)$ is a bounded function on $[0, d-1)$ so (4.14) implies (Hardy *et al* 1934, § 3.18) that $\log \phi$ is concave on $[0, d-1)$. The definition $\phi(d-1) = 1$ and (4.6) imply concavity on $[0, d-1]$.

Concavity implies continuity on $(0, d-1)$ and $\phi(0) \leqslant \phi(0+)$; continuity at 0 and $d-1$ follows from (4.10) and (4.6).

Lemma 4.5. If $0 \leqslant \gamma \leqslant d-1$, and c_n is a sequence of integers such that $0 \leqslant c_n \leqslant c_{\max}(n)$ and $\lim_{n\to\infty}(c_n/n) = \gamma$, then $\lim_{n\to\infty}(a_n(c_n))^{1/n}$ exists and equals $\phi(\gamma)$.

Proof. Suppose $0 \leqslant c_n \leqslant c_{\max}(n)$ and $\lim_{n\to\infty}(c_n/n) = \gamma$. The proof is in two parts, showing respectively

$$\phi(\gamma) \leqslant \liminf_{n\to\infty} a_n(c_n)^{1/n} \tag{4.15}$$

and

$$\limsup_{n\to\infty} a_n(c_n)^{1/n} \leqslant \phi(\gamma). \tag{4.16}$$

(*i*) First, for $\gamma = d-1$, (4.15) is immediate since $\phi(d-1) = 1$.

If $0 \leqslant \gamma < d-1$ let $\beta \in (\gamma, d-1)$. Then taking $c = c_n$ and $k = k_n \equiv \lceil \beta n \rceil - c_n$ (for n large enough so that $k_n > 0$) in inequality (2.11), we find

$$c_n^{k_n} a_n(\lceil \beta n \rceil) \leqslant [(d-1)n]^{k_n} a_n(c_n) \tag{4.17}$$

and thus

$$(c_n/n)^{k_n/n} a_n(\lceil \beta n \rceil)^{1/n} \leqslant (d-1)^{k_n/n} a_n(c_n)^{1/n}. \tag{4.18}$$

Letting $n \to \infty$, we find

$$\gamma^{\beta-\gamma}\phi(\beta) \leqslant (d-1)^{\beta-\gamma} \liminf_{n\to\infty} a_n(c_n)^{1/n}. \tag{4.19}$$

Let β decrease to γ; then, using lemma 4.3, equation (4.15) is true.

(*ii*) For $\gamma = 0$, equation (2.11) implies $a_n(c_n) \leqslant [(d-1)n]^{c_n} a_n(0)$ from which (4.16) follows.

If $0 < \gamma \leqslant d-1$ let $\alpha \in (0, \gamma)$. Then take $c = \lceil \alpha n \rceil$ and $k = j_n \equiv c_n - \lceil an \rceil$ (for n large enough so that $j_n > 0$) in inequality (2.11), so that

$$(\alpha n)^{j_n} a_n(c_n) \leqslant [(d-1)n]^{j_n} a_n(\lceil \alpha n \rceil) \tag{4.20}$$

and thus

$$\alpha^{j_n/n} a_n(c_n)^{1/n} \leqslant (d-1)^{j_n/n} a_n(\lceil \alpha n \rceil)^{1/n}. \tag{4.21}$$

Letting $n \to \infty$, and then letting α increase to γ, gives equation (4.16).

We close this section with some more properties of ϕ. Recall the definitions of $\lambda(\alpha)$ and α_0 from equations (1.7) and (1.10).

Theorem 3. (*i*) $\phi(\alpha) = \lambda(\alpha)$ for $\alpha \leqslant \alpha_0$.

(*ii*) $\alpha_0 \leqslant (d-1)/2$.

(*iii*) The (one-sided) derivatives of ϕ and $\log \phi$ at 0 are $+\infty$, and at $d-1$ are $-\infty$.

Proof. (*i*) Fix $\alpha \leqslant \alpha_0$. Let c_n be the c in $\{0, 1, \ldots, \lfloor \alpha n \rfloor\}$ which maximises $a_n(c)$. Then

$$a_n(c_n) \leqslant a_n(\alpha, \leqslant) \leqslant \alpha n a_n(c_n) \tag{4.22}$$

so we find from lemma 4.4 that

$$\phi(\alpha) \leqslant \lambda(\alpha) \leqslant \sup\{\phi(\gamma) \mid 0 \leqslant \gamma \leqslant \alpha\} \tag{4.23}$$

for all $\alpha \leqslant \alpha_0$. This implies that $\alpha_0 = \sup\{\alpha \mid \phi(\alpha) \leqslant \lambda\}$. Using lemma 4.4, we see that ϕ is strictly monotone increasing on $[0, \alpha_0]$. The result (*i*) now is immediate from equation (4.23).

(*ii*) The last statement of lemma 2.3 implies that ϕ is decreasing on $[(d-1)/2, d-1]$, from which (ii) follows.

(*iii*) For the right-hand derivative of $\log \phi(\alpha)$ (and then of $\phi(\alpha)$, via the chain rule) at $\alpha = 0$, divide (2.23) by α and let $\alpha \to 0$ (using part (*i*) above).

For the left-hand derivative at $\alpha = d-1$, putting $\beta = d-1$ in lemma 4.3 and taking logarithms gives

$$\alpha \log \alpha + (d-1-\alpha)\log(d-1-\alpha) - (d-1)\log(d-1) \geqslant \log \phi(d-1) - \log \phi(\alpha). \tag{4.24}$$

Divide by $d-1-\alpha$ and let α increase to $d-1$, obtaining $-\infty$ on the left.

5. Properties of the z transform

In this section we shall present some properties of the function $\Lambda(z)$, defined by equations (1.11) and (1.12). Since we are only interested in non-negative z, we will often write $z = \mathrm{e}^{\beta}$.

The first lemma includes some of the more elementary properties. We shall then prove the Legendre transform relationship, equation (1.13), which will then yield some additional results.

Lemma 5.1. The limit in equation (1.12) exists, so $\Lambda(z)$ is well defined for all $z \geqslant 0$, and has the following properties:

(*i*) $\Lambda(0) = \lambda_0$, $\Lambda(1) = \lambda$;

(*ii*) $\Lambda(z)$ is an increasing function of z;

(*iii*) $\log \Lambda(z) \leqslant \log \lambda + (d-1)\log z$ for all $z \geqslant 1$;

(*iv*) $\log \Lambda(\mathrm{e}^{\beta})$ is a convex function of β;

(*v*) $\Lambda(z)$ is continuous for $z \geqslant 0$.

Remark 1. Later in this section we shall show that $\Lambda(z)$ is actually strictly increasing.

Proof. The existence of the limit will be proven, as usual, by subadditivity arguments. A straightforward concatenation argument gives $\Sigma_{i=0}^{c} a_n(i) a_m(c-i) \leqslant a_{n+m}(c)$, from which we obtain

$$A_n(z) A_m(z) \leqslant A_{n+m}(z). \tag{5.1}$$

To complete the argument, we need upper bounds on $A_n(z)$. For $0 \leqslant z \leqslant 1$ we have $A_n(z) \leqslant a_n \leqslant \lambda^n$, and for $z \geqslant 1$ we have

$$A_n(z) \leqslant \sum_{c=0}^{(d-1)n-1} a_n z^n \leqslant (d-1) n \lambda^n z^{(d-1)n}. \tag{5.2}$$

Thus, for each $z \geq 0$, $(1/n) \log A_n(z)$ is a bounded function of n. With equation (5.1) this shows that the limit in equation (1.12) exists and is finite for all $z \geq 0$.

(i) is immediate and (ii) follows from the monotonicity of $A_n(z)$. (iii) follows from equation (5.2).

Hölder's inequality gives $A_n(\exp[\lambda\beta_1 + (1-\lambda)\beta_2]) \leq [A_n(e^{\beta_1})]^\lambda [A_n(e^{\beta_2})]^{1-\lambda}$ for $0 \leq \lambda \leq 1$, and real β_1, β_2; (iv) is an immediate consequence. Convexity implies continuity for all β, so to prove (v) we only need to investigate continuity at $z = 0$. First, observe that for $0 < z < 1$ and $0 < \alpha < \alpha_0$, $A_n(0) \leq A_n(z) \leq a_n(\alpha, \leq) + a_n z^{\alpha n}$ and therefore $\log \Lambda(z) \leq \max\{\log \lambda(\alpha), \alpha \log z + \log \lambda\}$. Letting z decrease to 0, we find

$$\Lambda(0) \leq \lim_{z \to 0^+} \Lambda(z) \leq \lambda(\alpha) \tag{5.3}$$

for all α in $(0, \alpha_0)$. Finally, $\Lambda(0) = \lambda(0)$, so (v) follows from equations (5.3) and (2.20).

Theorem 4. For all $z > 0$, $\log \Lambda(z) = \sup_{0 \leq \alpha \leq d-1}(\log \phi(\alpha) + \alpha \log z)$.

Proof. For any α in $[0, d-1)$, we have $\Lambda(z) \geq \lim_{n\to\infty}[a_n(\lceil \alpha n \rceil) z^{\lceil \alpha n \rceil}]^{1/n} = \phi(\alpha) z^\alpha$ and hence, by continuity at $\alpha = d-1$, we have

$$\log \Lambda(z) \geq \sup_{0 \leq \alpha \leq d-1} (\log \phi(\alpha) + \alpha \log z). \tag{5.4}$$

For the reverse inequality, fix z, and for each n, define c_n to be any integer c for which $a_n(c) z^c$ is maximised. Then $\Lambda(z) \leq \liminf_{n\to\infty}[(d-1) n a_n(c_n) z^{c_n}]^{1/n}$. There exists a ρ in $[0, d-1]$ which is the limit of some subsequence c_{n_k}/n_k; then lemma 4.5 implies $\Lambda(z) \leq \phi(\rho) z^\rho$. In conjunction with equation (5.4), this proves the theorem.

The next two corollaries are fundamental properties of Legendre transforms (e.g. Ellis 1985, theorem VI.5.3). First, we require one definition. Let $f: R \to (-\infty, +\infty]$ be a convex function. The subdifferential of f at the point y is defined to be $\partial f(y) = \{z \in R: f(x) - f(y) \geq z(x-y) \text{ for all } x\}$. Observe that if f is differentiable at y, then $\partial f(y) = \{f'(y)\}$. For example, if $f(x) = |x|$, then $\partial f(-3) = \{-1\}$ and $\partial f(0)$ is the interval $[-1, 1]$.

Corollary 1. For $0 \leq \alpha \leq d-1$, $\log \phi(\alpha) = \inf_{z>0}\{\log \Lambda(z) - \alpha \log z\}$.

Corollary 2. $\log \Lambda(z) = \log \phi(\alpha) + \alpha \log z$ if and only if $\log z$ is in the subdifferential of $-\log \phi(\alpha)$.

Corollary 3. $\Lambda(z)$ is strictly increasing for $z \geq 0$.

Proof. Let $H(\beta) = \log \Lambda(e^\beta)$; it suffices to prove that $H(\beta)$ is strictly increasing.

For any finite β, the function $\log \phi(\alpha) + \alpha\beta$ is not maximised at $\alpha = 0$ (by (iii) of theorem 3), so $H(\beta) > \log \phi(0)$ for all finite β. But lemma 5.1 tells us that $\lim_{\beta \to -\infty} H(\beta) = \log \phi(0)$ and that $H(\beta)$ is convex, so it follows that $H(\beta)$ is strictly increasing.

Corollary 4. $\lim_{z\to\infty} \log \Lambda(z)/\log z = d-1$.

Proof. For each z, let $\alpha(z)$ be the smallest α such that $\log z$ is in the subdifferential of $-\log \phi(\alpha)$; then $\log \Lambda(z)/\log z = \log \phi(\alpha(z))/\log z] + \alpha(z)$. Part (*iii*) of theorem 3 implies that $\lim_{z\to\infty} \alpha(z) = d-1$, so the corollary is proven.

6. Discussion

Although there has been renewed interest in the lattice animal problem since Lubensky and Isaacson (1979) proposed this as a model of branched polymers, there have been relatively few rigorous results. Of course, there has been substantial progress in the rigorous theory of percolation (Kesten 1982) and lattice animals are closely related to percolation clusters. However, the associated weights are different in the two problems. The primary purpose of this paper is to prove a set of rigorous results on the animal problem.

The results of this paper are summarised schematically in figure 2. The general properties obtained by us for the three functions $\lambda(\alpha)$, $\phi(\alpha)$ and $\Lambda(z)$ are illustrated and the relationships between the three functions can be seen.

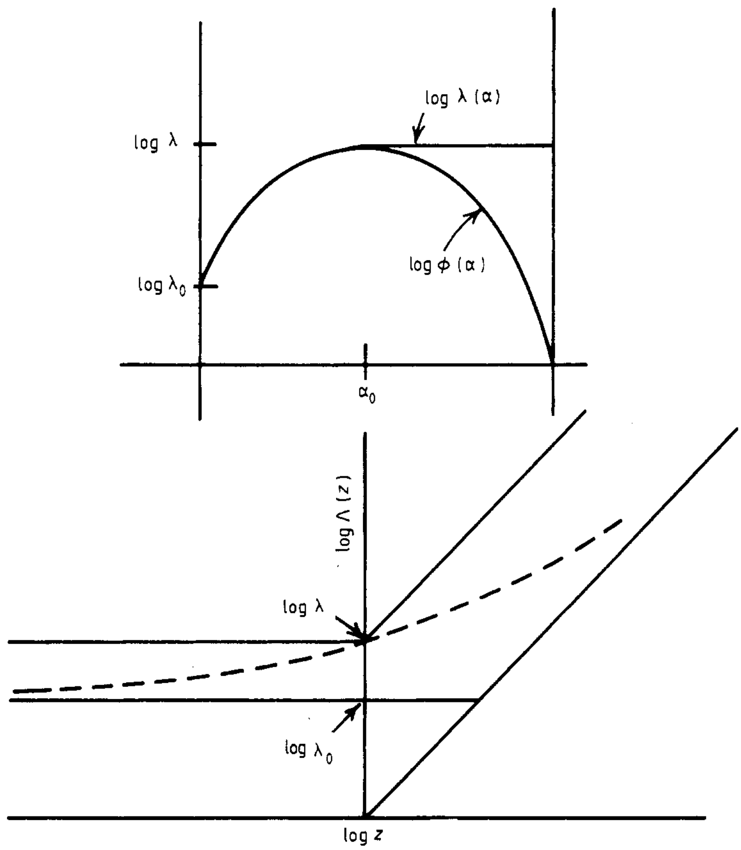

Figure 2. Expected behaviour of the functions $\lambda(\alpha)$, $\phi(\alpha)$ and $\Lambda(z)$.

One motivation for this work is the connection with the collapse transition in branched polymers (Dickman and Schieve 1986, Lam 1987). In this regard many problems remain to be addressed. Is the function $\Lambda(z)$ analytic? If not, there is a collapse transition. Is $\phi(\alpha)$ strictly concave? If not, then $\Lambda(z)$ is not differentiable and there is a first-order phase transition. In particular, is the maximum value in $\phi(\alpha)$ attained at a unique point? If not, there is a collapse transition, but at $z=1$, i.e. at 'infinite temperature'. These are difficult problems which we feel are worthy of attention.

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